Interlaminar Stress Continuity Theory for Laminated Composite Analysis

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Introduction

BOTH the interlaminar shear stresses and the interlaminar normal stress are important stress components in composite failure analysis. It has been concluded from many investigations that the high interlaminar stresses can cause damage on the composite interface. For example, the interlaminar shear stress on the free edge of angle-ply laminates can cause edge delamination,[1] as can the interlaminar normal stress in the cross-ply laminates.[2]

Although the interlaminar stresses are very important, their magnitudes are much lower than the in-plane components, and they are very difficult to obtain. Frequently, in composite analysis, only the displacements and in-plane stresses are of interest.[3-6] However, if the interlaminar stresses are of concern, equilibrium equations are used as a post-processing technique for stress recovery.[7] Although this indirect technique can give accurate interlaminar stresses, there is concern for using this technique in general application, e.g., it may become inefficient for structures with highly irregular geometry.

Another way to find the interlaminar stresses is to use multiple-layer theory in conjunction with the stress continuity equations on the composite interface. The hybrid-stress finite element method presented by Mau et al.[8] satisfies the continuity equations exactly. The interlaminar stresses can be obtained directly from the assumed stress functions. To reduce the numerical complexity, Spilker[9] used an alternate method to obtain the interlaminar stresses. Since the technique satisfies the continuity equations in an approximate sense, the results are not as accurate as those of Mau et al.

In view of the importance of satisfying the continuity equations exactly on the composite interface and obtaining the interlaminar stresses directly from the constitutive equations, Lu and Liu[10] have presented an interlaminar shear stress continuity theory. Their theory can be viewed as a refined form developed from the techniques presented by Di Sciuva,[3] Hinrichsen and Palazotto,[4] Toledano and Murakami,[5] and Reddy.[6] It is the only one, however, that can obtain the interlaminar shear stresses directly from constitutive equations. Numerical results show that their theory achieves excellent results for all displacement components and some stresses. However, due to the negligence of the deformation in the thickness direction, the interlaminar normal stress cannot be calculated from the constitutive equations directly. In addition, a small discrepancy between their results and Pagano's elasticity solutions[11] in the interlaminar shear stress for composite laminates with small aspect ratios has also been reported.

Because of the important role of the interlaminar stresses in the composite delamination analysis, it is the objective of this study to present a theory that can accurately predict the interlaminar stresses. This theory should satisfy the continuity equations for both interlaminar shear stress and interlaminar normal stress. A finite element method based on the principle of minimum potential energy is used for numerical analysis.

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Displacement Field

The derivation of the new theory is from the displacement approach. To begin with, only the composite laminates under cylindrical bending are considered. Therefore, it is assumed that the displacement field of an n-layer composite laminate, as shown in Fig. 1, can be expressed by the following equations:

$$u(x,z) = u_0(x) + \sum_{i=1}^{n} (U_{i-1}\phi_1^{(i)} + \hat{S}_{2i-2}\phi_2^{(i)} + U_i\phi_3^{(i)} + \hat{S}_{2i-1}\phi_4^{(i)}) \quad (1a)$$

$$w(x,z) = w_0(x) + \sum_{i=1}^{n} (W_{i-1}\phi_1^{(i)} + \hat{R}_{2i-2}\phi_2^{(i)} + W_i\phi_3^{(i)} + \hat{R}_{2i-1}\phi_4^{(i)}) \quad (1b)$$

In the preceding equations, u_0 and w_0 are the displacements on the midplane of the composite laminate in the axial and thickness direction, respectively. U and W which vanish at the midplane, represent the interfacial translation variables, respectively, whereas $\hat{S}$ and $\hat{R}$ represent the interfacial rotation variables. The subscripts of these variables indicate the interfaces of interest, as shown in Fig. 1.

In order to observe the parabolic distribution of the interlaminar shear stress through the thickness of each layer and to satisfy the continuity of interlaminar stresses between the composite layers, a cubic interpolation function, namely, Hermite cubic shape function, is used. Its components are defined as follows:

$$\phi_1^{(i)} = 1 - \frac{3(z-z_{i-1})^2}{h_i^2} + \frac{2(z-z_{i-1})^3}{h_i^3}$$

$$\phi_2^{(i)} = (z-z_{i-1})\left(1 - \frac{z-z_{i-1}}{h_i}\right)^2$$

$$\phi_3^{(i)} = \frac{3(z-z_{i-1})^2}{h_i^2} - \frac{2(z-z_{i-1})^3}{h_i^3} \qquad z_{i-1} \le z \le z_i \quad (2a)$$

$$\phi_4^{(i)} = (z-z_{i-1})\left[\frac{(z-z_{i-1})^2}{h_i^2} - \frac{z-z_{i-1}}{h_i}\right]$$

$$\phi_1^{(i)} = \phi_2^{(i)} = \phi_3^{(i)} = \phi_4^{(i)} = 0, \qquad z < z_{i-1} \text{ or } z > z_i \quad (2b)$$

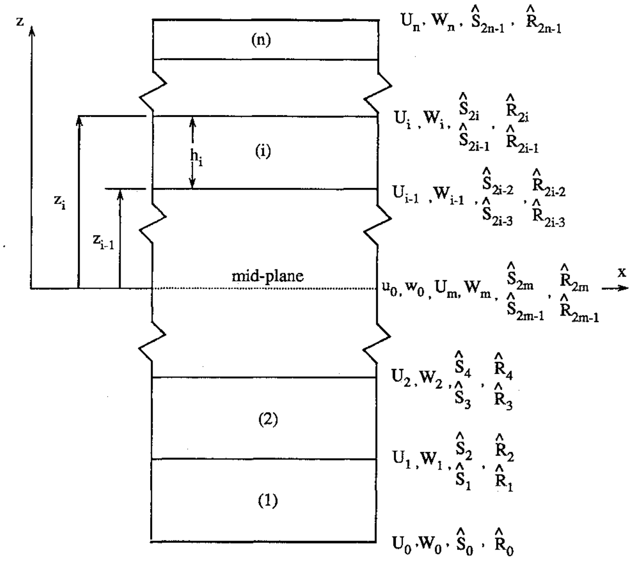

Fig. 1 Displacement components and coordinate system.

The superscript (i) is to indicate the layer number. The coordinates z are measured from the midplane, whereas h_i is the thickness of the ith layer. The definitions for some notations can also be found in Fig. 1.

In this study, small deformation and deflection are assumed; i.e., the relationships between strains and displacements are linear:

$$\epsilon_x = \frac{\partial u}{\partial x} \tag{3a}$$

$$\epsilon_z = \frac{\partial w}{\partial z} \tag{3b}$$

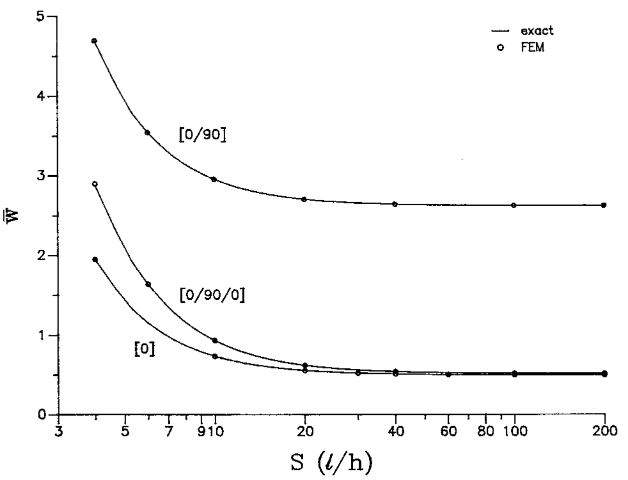

Fig. 2 Normalized lateral displacements at the midplane and midspan of the simply supported beams under sinusoidal loading.

$$2\epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{3c}$$

The composite laminates are assumed to be made of orthotropic layers. For composite laminates under cylindrical bending, the following constitutive equations for plane strain are used[12]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \sigma_{xz} \end{Bmatrix} = \begin{pmatrix} Q_{11} & Q_{13} & 0 \\ Q_{13} & Q_{33} & 0 \\ 0 & 0 & Q_{55} \end{pmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_z \\ 2\epsilon_{xz} \end{Bmatrix} \tag{4}$$

By assigning a single value for translation at each interface, the continuity of translations through the thickness is automatically satisfied. Since the major theme of this study is to investigate the role of interlaminar stress continuity on the interlaminar stress analysis, the continuity equations for both interlaminar normal stress and interlaminar shear stress should be satisfied; i.e.,

$$\sigma_{xz}^{(i)}\Big|_{z=z_i} = \sigma_{xz}^{(i+1)}\Big|_{z=z_i}, \qquad \sigma_z^{(i)}\Big|_{z=z_i} = \sigma_z^{(i+1)}\Big|_{z=z_i} \tag{5}$$

in which $i = 1, 2, \ldots, n-1$.

Although the interlaminar stresses are continuous across the composite interface, the interlaminar strains are not. This is the reason why two interfacial rotation variables are assigned at an interface in Eqs. (1a) and (1b). However, to meet the requirements of the interlaminar stress continuity conditions, the displacement field is substituted into the strain-displacement relations, then into the constitutive equations. By employing the continuity equations, the following two relations can be concluded; where $i = 1, 2, \ldots, n-1$:

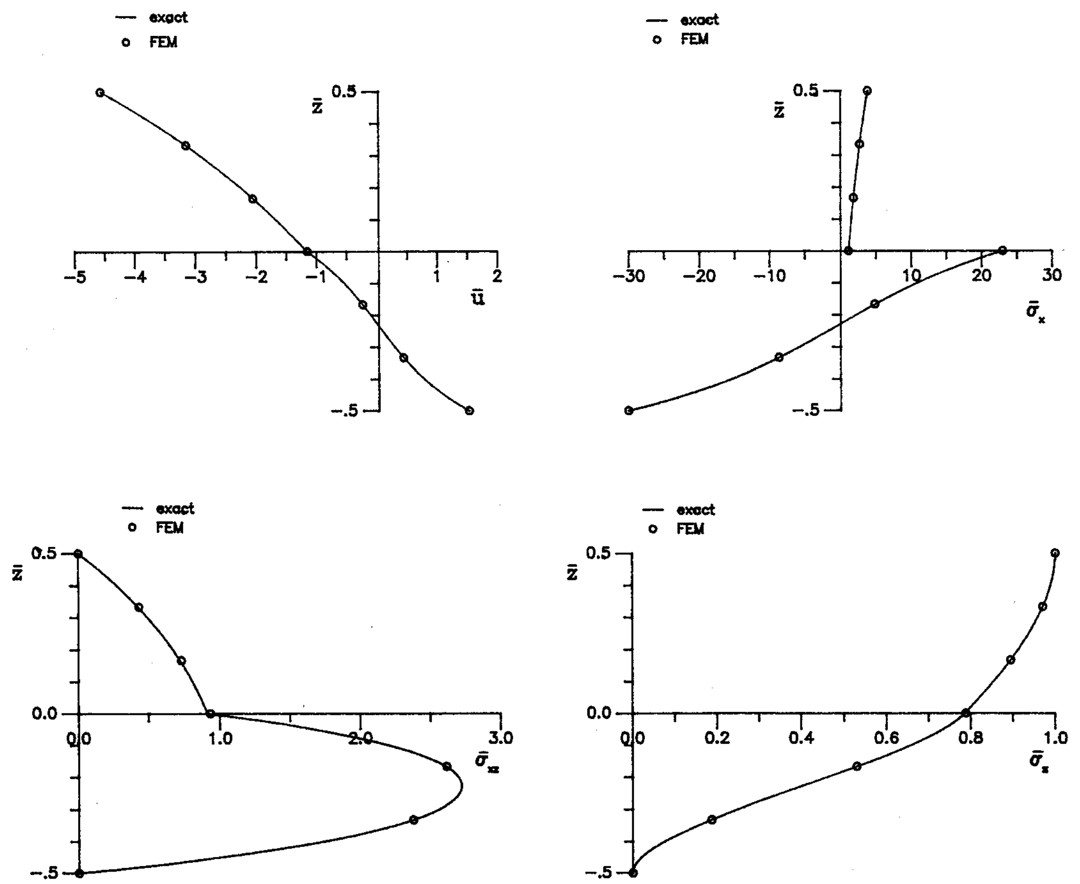

Fig. 3 Normalized displacement and stresses of simply supported (0/90) beam with $S = 4$.

$$\hat{S}_{2i-1} = \frac{Q_{55}^{(i+1)}}{Q_{55}^{(i)}}\hat{S}_{2i} + \left(\frac{Q_{55}^{(i+1)}}{Q_{55}^{(i)}} - 1\right)\frac{dw_0}{dx} + \left(\frac{Q_{55}^{(i+1)}}{Q_{55}^{(i)}} - 1\right)\frac{dW_i}{dx} \tag{6a}$$

$$\hat{R}_{2i-1} = \frac{Q_{33}^{(i+1)}}{Q_{33}^{(i)}}\hat{R}_{2i} + \left(\frac{Q_{13}^{(i+1)} - Q_{13}^{(i)}}{Q_{33}^{(i)}}\right)\frac{du_0}{dx} + \left(\frac{Q_{13}^{(i+1)} - Q_{13}^{(i)}}{Q_{33}^{(i)}}\right)\frac{dU_i}{dx} \tag{6b}$$

These relations imply that one of the two rotation variables at an interface is not independent, although they are not identical. Therefore, only one independent rotation variable is required at each interface. The new variables are then assigned as follows:

$$S_i = \hat{S}_{2i}, \qquad i = 0, 1, 2, \ldots, n-1; \qquad S_n = \hat{S}_{2n-1}$$

$$R_i = \hat{R}_{2i}, \qquad i = 0, 1, 2, \ldots, n-1; \qquad R_n = \hat{R}_{2n-1}$$

With the new notations, the displacement field can be rewritten in a form with fewer variables, i.e.,

$$u(x,z) = u_0(x) + \sum_{i=1}^{n}\left(U_{i-1}\Phi_1^{(i)} + S_{i-1}\Phi_2^{(i)} + U_i\Phi_3^{(i)} + S_i\Phi_4^{(i)} + \frac{dw_0}{dx}\Phi_5^{(i)} + \frac{dW_i}{dx}\Phi_5^{(i)}\right) \tag{7a}$$

$$w(x,z) = w_0(x) + \sum_{i=1}^{n}\left(W_{i-1}\Phi_1^{(i)} + R_{i-1}\Phi_2^{(i)} + W_i\Phi_3^{(i)} + R_i\Phi_6^{(i)} + \frac{du_0}{dx}\Phi_7^{(i)} + \frac{dU_i}{dx}\Phi_7^{(i)}\right) \tag{7b}$$

where Φ are the new interpolation functions, which are functions of ϕ and Q.

Finite Element Formulation

To formulate the governing equations, the principle of minimum potential energy is used. The total potential energy of the laminated composite subjected to a transverse loading $q(x)$ on the top surface can be expressed as follows:

$$\Pi = \frac{1}{2}\int_0^{\ell}\int_{-h/2}^{h/2}[\sigma_x\epsilon_x + \sigma_z\epsilon_z + \sigma_{xz}(2\epsilon_{xz})]\,dz\,dx - \int_0^{\ell}[qw\,|_{z=h/2}]\,dx \tag{8}$$

The stresses can be substituted by the strains using Eq. (4), and the strains in the ith layer can be obtained in terms of displacement components by combining Eqs. (3) and (7). Carrying out the inner integration of the first term on the right-hand side of Eq. (8), the total potential energy can be expressed in terms of interfacial displacement variables introduced in Eqs. (7). It should be noted that the vanished midplane displacement components and the traction boundary conditions on the top and bottom surfaces of the laminate can be used to reduce the total independent interfacial variables.

Upon our completion of the derivation of the total potential energy in terms of the interfacial variables in the thickness direction, the formulation for the finite element equation can proceed as an ordinary procedure. In this study, Hermite cubic shape functions are also used for the interpolations of all interfacial variables in the x direction. By using the principle of minimum potential energy, the element stiffness matrix and force vector can be derived.

Numerical Results and Discussions

In order to verify the new theory, numerical solutions for composite laminates subjected to sinusoidal loading are examined. The results are compared with those of Pagano's solutions[11] for accuracy and convergence studies.

Three types of composite laminates are studied. They are [0] unidirectional, [0/90] asymmetric, and [0/90/0] cross-ply laminates. The elastic constants are exactly the same as those used in Ref. 11, i.e., $E_L = 172$ GPa, $E_T = 6.9$ GPa, $G_{LT} = 3.5$ GPa, $G_{TT} = 1.4$ GPa, and $\nu_{LT} = \nu_{TT} = 0.25$. The deflection and the interlaminar normal stress in the middle of the span and the interlaminar shear stress at the free edge of the laminates are of interest. The normalized values are defined exactly the same as those in Ref. 11.

To verify the feasibility of using the new theory for both thin and thick composite laminates, the normalized deflections at the midplane and midspan are presented in Fig. 2. It is in a logarithmic scale with the aspect ratio ranging from 4 to 200 for all three laminates. The results show that the new theory can be used for both thin and thick composite laminates analysis.

Since the main advantage of the present theory is the calculation of all stress components directly from the constitutive equations, the accuracy of the result should be demonstrated. Herein, the results of normalized inplane displacement and three stresses of a [0/90] asymmetric laminate using ten elements and six layers are presented in Fig. 3. Apparently, excellent agreement for all displacement and stresses is observed.

References

[1]Pipes, R. B., and Pagano, N. J., "Interlaminar Stresses in Composite Laminates Under Uniform Axial Loading," *Journal of Composite Materials*, Vol. 4, No. 4, 1970, pp. 538–548.

[2]Rybicki, E. F., "Approximate Three-Dimensional Solutions for Symmetric Laminates Under Inplane Loading," *Journal of Composite Materials*, Vol. 5, No. 2, 1971, pp. 354–360.

[3]DiSciuva, M., "Development of An Anisotropic Multilayered Shear-Deformable Rectangular Plate Element," *Computers and Structures*, Vol. 21, No. 4, 1985, pp. 789–796.

[4]Hinrichsen, R. L., and Palazotto, A. N., "Nonlinear Finite Element Analysis of Thick Composite Plates Using Cubic Spline Functions," *AIAA Journal*, Vol. 24, No. 11, 1986, pp. 1836–1842.

[5]Toledano, A., and Murakami, H., "A Composite Plate Theory for Arbitrary Laminate Configurations," *Journal of Applied Mechanics*, Vol. 54, 1987, pp. 181–189.

[6]Reddy, J. N., "A Generalization of Two-dimensional Theories of Laminated Composite Plates," *Communications in Applied Numerical Methods*, Vol. 3, 1987, pp. 173–180.

[7]Lo, K. H., Christensen, R. M., and Wu, E. M., "Stress Solution Determination for High Order Plate Theory," *International Journal of Solids and Structures*, Vol. 14, 1978, pp. 655–662.

[8]Mau, S. T., Tong, P., and Pian, T. H. H., "Finite Element Solutions for Laminated Thick Plates," *Journal of Composite Materials*, Vol. 6, No. 2, 1972, pp. 304–311.

[9]Spilker, R. L., "A Hybrid-stress Finite-element Formulation for Thick Multilayer Laminates," *Computers and Structures*, Vol. 11, No. 3, 1980, pp. 507–514.

[10]Lu, X., and Liu, D., "An Interlaminar Shear Stress Continuity Theory," *Proceedings of the Fifth Technical Conference of the American Society for Composites*, Technomic, Lancaster, PA, 1990, pp. 479–483.

[11]Pagano, N. J., "Exact Solutions for Composite Laminates in Cylindrical Bending," *Journal of Composite Materials*, Vol. 3, No. 3, 1969, pp. 398–411.

[12]Vinson, J. R., and Sierakowski, R. L., *The Behavior of Structures Composed of Composite Materials*, Martinus Nijhoff, Dordrecht, The Netherlands, 1986, Chap. 2.